JOURNAL OF FINANCIAL INTERMEDIATION **3**, 204–212 (1994)

On the Equivalence of Noise Trader and Hedger Models in Market Microstructure*

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Received July 14, 1993

It is shown that the models of Spiegel and Subrahmanyam (1992, *Rev. Finan. Stud.* 5(2), 307–329) and Kyle (1985, *Econometrica* **53**, 1315–1335) are equivalent in the following sense: the equilibrium values of market depth, the expected total trading volume and the expected price level are the same in the two models. Equivalence exists whenever the uniformed traders hedge all of their endowments of risky shares. This occurs under two sets of parameter configurations. In both cases, the linear equilibrium in the hedger model always exists. *Journal of Economic Literature* Classification Numbers: G12, G14, D82. © 1994 Academic Press, Inc.

Spiegel and Subrahmanyam (1992) extended the framework of Kyle (1985) by modeling noise traders as risk-averse investors who trade to hedge their endowment of shares of the risky asset. Their paper shows that several comparative static results in Kyle (1985) are upset by their generalization and they conclude by stating that "postulating the existence of noise traders with exogenous demands, therefore, is not solely a convenient modeling device . . . it is an assumption whose relaxation alters many of the basic results."

This note shows that under certain conditions the Spiegel and Subrahmanyam (1992) model is equivalent to Kyle's (1985) model with noise trading.[1] Specifically, if each uniformed trader hedges all of her endowment of risky shares, the two models are observationally equivalent.

* I thank three anonymous referees and seminar participants at the University of Illinois for their comments.

[1] All citations to Kyle's (1985) model refer to the static solution only.

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Equivalence occurs for two sets of parameter configurations. In both cases, the linear equilibrium in the hedger model always exists—just as in Kyle's (1985) model.

Equivalence is defined in the following sense: the equilibrium value of market depth, the expected total trading volumes, and the expected price levels are exactly the same.[2] Note that equivalence is defined with respect to aggregate market variables,[3] which a casual market observer might reasonably be expected to know or find out.[4] If two markets are equivalent in the above sense, such an observer should, on average, be indifferent as to which he would prefer to trade in. In other words, the apparently separate models are really describing the same underlying market.

What is the intuition behind this equivalence result? In Spiegel and Subrahmanyam's (1992) model, each uninformed trader j is risk-averse and endowed with w^j shares of the risky asset. Equivalence occurs whenever, in equilibrium, each uninformed trader hedges perfectly—i.e., sells (buys) exactly w^j shares if w^j is positive (negative). In this case, although noise trading is endogenous, the equilibrium distribution of aggregate noise trading is simply the exogenously given distribution of aggregate endowments. If the "noisiness" is the same in the two models, these two distributions are identical and the equilibrium outcomes do not differ.

There are two reasons that perfect hedging may not occur in general.[5] First, suppose there are an infinite number of hedgers, but that the coefficient of absolute risk aversion is finite. Then each uninformed trader will underhedge—the hedge coefficient will be less than 1 in absolute value. The reason is that each hedger's certainty equivalent payoff function will assign finite weight to both her expected payoffs and the variance of these payoffs. As the risk aversion coefficient goes to infinity, in the limit the hedger will be concerned only with the variance of her profits and will hedge all of her endowment.

[2] The actual total trading volumes and price levels are not comparable since these depend on the actual volume of noise trading, which is not a choice variable in Kyle (1985). Further, the expectations are of the absolute values of these variables.

[3] In particular, the composition of the total trading volume between informed and uninformed trading is difficult to discern and so it is not required that the equilibrium values of these variables be equated as well. In models where there exists a distinct group of uninformed traders, this distinction is not important because equivalence between any two models in the sense defined here will automatically imply that the expected informed and uninformed trading volumes are also equal. In models where the same investor can trade for informational as well as, say, for risk-sharing purposes, the distinction *is* important and the expected informed and uninformed trading volumes need not be equal. See Sarkar (1992).

[4] Market depth is not directly observable, but is easily inferred as the ratio of the expected price level to the expected total trading volume.

[5] See (4) of Section IIB and the discussion thereafter for a formal representation of these arguments.

Next, suppose that the risk aversion coefficients is infinite but the number of hedgers is finite. In this case, each uninformed trader will overhedge—the hedge coefficient will be greater than 1 in absolute value. The reason is as given in Spiegel and Subrahmanyam (1992): each hedger reduces the variance of her payoffs by increasing the covariance between the price and the fundamental value of the asset (equivalently, the variance of the price). When the number of hedgers is small, each of them must trade a large amount to induce enough variability in the price. As the number of hedgers goes to infinity, the contribution of each hedger's trades to the total variability of the order flow is negligible. Again, perfect hedging is optimal.

Equivalence between the hedger and the noise trader models is derived for two separate cases. In both, the linear equilibrium always exists, as in Kyle's (1985) model. In the first case the number of hedgers is finite, and the hedger's risk-aversion parameter must have a specific value which depends upon the variances of the asset value and of the hedgers' endowments as well as upon the amount of information in the market. In the second case, equivalence is derived for a limit economy where k, the number of uninformed investors, and R, the coefficient of absolute risk aversion in each hedger's utility function, go to infinity. If Σ_w is the variance of the hedger's endowment of risky shares, equivalence holds if $k\Sigma_w$ tends to a finite number (which implicitly requires that $\Sigma_w \rightarrow 0$ as $k \rightarrow \infty$).

How sensitive or robust are the properties of Kyle's (1985) equilibrium when we drop the assumption that noise trading is exogenous? As Spiegel and Subrahmanyam (1992) have shown, several of the comparative static results change when noise trading is endogenized in a natural way. On the other hand, these results demonstrate that, in markets where the volume of hedging-motivated trades is not too high or low (there is no over- or underhedging), the equilibrium outcome of Kyle's (1985) model is a good approximation of reality.[6] Empirically, this conclusion can be tested by checking whether the direction of the comparative static results (the effect of adverse selection measures on liquidity, for example) are different between markets (stocks versus futures, for example) or at different times in the same market (changes in liquidity in the stock market before and after the introduction of futures or options markets in the U.S., for example).

The remainder of this paper is organized as follows. Section I lays out a generalized version of Kyle (1985) and restates the results of Spiegel and Subrahmanyam (1992). Section II shows the equivalence of these two

[6] In related literature, Krishnan (1992) has shown that when there are just two states of uncertainty, the pricing rule in Kyle's (1985) model can be interpreted as the bid–ask prices in Glosten and Milgrom (1985).

models for both the finite and the limit economy. Section III concludes. All proofs are in the appendix.

I. UNINFORMED TRADERS AS HEDGERS

In Section A, a generalized version of Kyle (1985) is described with multiple informed traders and imperfect information. The extension of Kyle's (1985) single-period model to the case of multiple informed traders was earlier presented in Admati and Pfleiderer (1988) and Subrahmanyam (1991), among others. This is followed in Section B with a statement of the model of Spiegel and Subrahmanyam (1992).

A. *The Noise Trading Model*

Consider a market in which a single risky asset with an unknown liquidation value v is traded. There is a group of m informed traders each of whom receives, prior to trading, signals s^i about the unknown value v. The signals are of the form $s^i = v + e^i$, $i = 1, \ldots, m$. In addition, there is a group of uniformed noise traders who trade for liquidity reasons.

Each informed trader $i = 1, \ldots, m$ submits a market order x^i to a market maker. The noise traders also collectively submits market orders worth u_n to the same market maker for execution. The latter then fixes a single price p_n at which she will execute the total order flow $y_n = x_n + u_n$, where x_n is the combined market orders of the informed group. The market maker is assumed to be risk-neutral and competitive. Conditional on observing y_n, she earns zero expected profits. Finally, the random variables, v, u_n and e^i, $i = 1, \ldots, m$, are independent of each other and normally distributed with zero mean and finite variances Σ_v, Σ_u, and Σ_e, respectively.

Suppose that each informed investor i conjectures that the market maker's pricing rule is linear in the total order flow: $p_n = \Gamma_n y_n$, where $\Gamma_n = \mathrm{Cov}(v, y_n)/\mathrm{Var}(y_n)$ is the market-depth parameter. Suppose also that all informed investors follow linear trading rules $x^i = A_n s^i$, $i = 1, \ldots, m$. Each informed trader i chooses x^i to maximize his conditional expected profits $E(\pi_n^i|s^i)$, where $\pi_n^i = (v - \Gamma_n x^i - \Gamma_n \sum_{j \neq i} x^j - \Gamma_n u_n)x^i$. Define $t = \Sigma_v/(\Sigma_v + \Sigma_e)$ and note that $0 \leq t \leq 1$, where t is a measure of the unconditional precision of s^i, $i = 1, \ldots, m$. Also, define $Q = [1 + t(m - 1)]$. Then Lemma 1 describes the linear equilibrium solution for the noise trader model.

LEMMA 1. *In the noise trader model, each informed trader* $i = 1, \ldots,$ m *trades* $x^i = A_n s^i$ *and the price is* $p_n = \Gamma_n y_n$, *where* $A_n = t/\Gamma_n(1 + Q)$ *and* $\Gamma_n = \sqrt{mt\Sigma_v}/(1 + Q)\sqrt{\Sigma_u}$.

Proof. See the proof of Lemma 1 in Admati and Pfleiderer (1988).

Next, the basic model is extended to allow for rational behavior by uninformed traders.

B. *The Model with Uninformed Traders as Rational Hedgers*

There are k risk-averse uninformed traders (hedgers) who trade for purely high risk-sharing reasons. The development of the model here follows Spiegel and Subrahmanyam (1992). Each hedger j has random endowment w^j, which is assumed to be normally distributed with mean zero and variance Σ_w. Then w^j, $j = 1, \ldots, k$ are independent of each other and all other random variables in the model. All hedgers have negative exponential utility functions with risk-aversion parameter R.

Suppose that all hedgers submit market orders u^j to the market maker and follow linear trading rules of the form $u^j = Dw^j, j = 1, \ldots, k$. Let the total uninformed trades be $u_h = \sum_{j=1}^{k} u^j$. If π_h^j is the profit of the jth hedger, then u^j is chosen to maximize her utility or certainty-equivalent profits $V_h^j = E(\pi_h^j|w^j) - R\,\mathrm{Var}(\pi_h^j|w^j)/2$. The informed traders maximization problem remains the same as before, since each w^j is independent of v.[7]

Market depth is now positively related to the magnitude of the hedge coefficient, D (since this increases the variance of the total order flow), and to the risk aversion parameter, R. Further, the equilibrium $D < 0$, since the marginal utility of the hedgers from a purchase (sale) is negative if endowments are positive (negative).

Let x_h and Γ_h be the total informed trading volume and the market depth parameter in the hedger model, respectively. Lemma 2 (which is identical to Proposition 1 in Spiegel and Subrahmanyam (1992))[8] describes the equilibrium. Equilibrium exists if the amount of risk-aversion and noise in the market exceeds the amount of information available.

LEMMA 2. *An equilibrium to the hedger model exists if* R satisfies

$$R\,\Sigma_v(2 - t) > \frac{2\sqrt{mt\Sigma_v}}{\sqrt{k\Sigma_w}}. \tag{1}$$

In equilibrium, each hedger $j = 1, \ldots, k$ *trades* $u^j = Dw^j$, *where* $D < 0$, *market depth is* $1/\Gamma_h$, *and,*

[7] Of course, the actual informed trading volumes will be different since market depth will be different, in general.

[8] In the expression of D (γ in their paper) in Spiegel and Subrahmanyam (1992), there should be a minus sign in front. This is missing in their paper due to a typographic error.

$$\Gamma_h D = -\frac{\sqrt{mt\Sigma_v}}{(1+Q)\sqrt{k\Sigma_w}} \tag{2}$$

$$D = \frac{2\sqrt{mt\Sigma_v/k\Sigma_w} - R\Sigma_v(2-t)}{R\Sigma_v(2-t) - R\Sigma_v mt/k(1+Q)}. \tag{3}$$

Each informed trader $i = 1, \ldots, m$ *trades* $x^i = A_h s^i$, *where* $A_h = t/\Gamma_h(1+Q)$. *The price is* $p_h = \Gamma_h y_h$, *where* $y_h = x_h + u_h$.

Proof. See the proof of Proposition 1 in Spiegel and Subrahmanyam (1992).

Comparing Lemmas 1 and 2, it is apparent that the two models of uninformed trading differ essentially along two dimensions: the magnitude of the hedge coefficient D, and the amount of noise, which is Σ_u in one case and $k\Sigma_w$ in the other. These two factors, in turn, determine the differences in equilibrium market depth, expected prices and expected trading volumes. If $-D\sqrt{k\Sigma_w} = \sqrt{\Sigma_u}$, then the models are clearly equivalent. In the next section, two sets of parameter values are considered such that this equality holds.

II. THE EQUIVALENCE OF THE TWO MODELS

In this section, it is shown that there exist conditions under which the equilibrium solutions of the two models are equivalent in the following sense: market depth, the expected total trading volumes, and the expected price levels have the same equilibrium values. The expected trading volumes in market i = n, h are computed as $E(|x_i|) + E(|u_i|)$, where $|x_i|$ and $|y_i|$ are the absolute values of the net informed trading volume and the net uninformed trading volume, respectively. In the noise trader model, $E(|u_n|) = \sqrt{\Sigma_u}/\sqrt{2\pi}$ and in the hedger model, $E(|u_h|) = |D|\sqrt{k\Sigma_w}/\sqrt{w\pi}$. Similarly, $E(|x_i|) = A_i\sqrt{m\Sigma_s Q}/\sqrt{2\pi}$, i = n, h, where Q was defined earlier.

Even if the definition of equivalence is satisfied, the two models share the same equilibrium solution on average only since it is the expected and not the actual total trading volumes (and, by implication, the actual prices) which are being equated. There is no sensible way that the actual total trading volumes can be equated since this would require the actual uninformed trading volumes to be equated.

A. *Equivalence in a Finite Economy*

Suppose that the amount of noise in the two economies are pegged at the same level, i.e., $\Sigma_u = k\Sigma_w$. Then, from the discussion following Lemma 2, if $D = -1$ (uninformed traders hedge perfectly), the two models are equivalent. For $D = -1$ to be satisfied the risk-aversion parameter R must satisfy a particular value given in Proposition 1. Further, at this value of R, (1) is always satisfied and so the linear equilibrium always exists in the hedger model.

PROPOSITION 1. *Suppose that* $\Sigma_u = k\Sigma_w$. *Then if* $R = 2(1 + Q)\sqrt{k}/\sqrt{mt\Sigma_v\Sigma_w}$, *equilibrium always exists in the hedger model and the two models of uninformed trading are equivalent.*

Given the value of R specified in Proposition 1, $D = -1$ and each uninformed trader hedges all of her endowment in equilibrium. Uninformed traders as a group trade (the negative of) $w_h = \sum_{j=1}^{k} w^j$. Thus, the equilibrium behavior of the hedgers mimic that of the noise traders. The additional assumption $k\Sigma_w = \Sigma_u$ amounts to a scaling of these variances ensuring that the total amount of noise in the two models is the same.

B. *Equivalence in a Limit Economy*

Rewrite the expression for D from (3) as follows:

$$D = \left(\frac{2}{R}\frac{\sqrt{mt}}{\sqrt{k\Sigma_w\Sigma_v}} - (2 - t)\right)\Big/\left((2 - t) - \frac{mt}{k(1 + Q)}\right). \tag{4}$$

Suppose $k \to \infty$ and $R \to \infty$ while $k\Sigma_w \to \Sigma_u$. Then $D \to -1$, $\Gamma_h \to \Gamma_n$ and the two models are equivalent. Note that if R is finite while the other two conditions hold, the absolute value of $D < 1$. If k is finite but the other two conditions hold, then $|D| > 1$. Thus, the limiting result requires that both $k \to \infty$ and $R \to \infty$. The condition $k\Sigma_w \to \Sigma_u$ (finite) is necessary to prevent the market from being infinitely liquid. Note also that, in the limit, the existence condition (1) always holds because the left-hand side goes to infinity while the right-hand side remains bounded. This leads to the following result.

PROPOSITION 2. *Suppose* $k \to \infty$, $R \to \infty$ *and* $k\Sigma_w \to \Sigma_u$. *Then, the linear equilibrium in the hedger model always exists and is equivalent to Kyle's* (1985) *equilibrium solution.*

Proof. Follows directly from (1) and (4).

III. CONCLUSION

This paper has examined the robustness of the Kyle (1985) model with respect to the noise trader assumption by comparing it to the alternative representation of Spiegel and Subrahmanyam (1992) where the uninformed traders behave as rational agents but the batch market mechanism for price determination is left intact. Here, the uninformed traders are viewed as risk-averse investors who trade to hedge their endowment of risky shares. It was shown that when the uninformed traders hedge all of their endowments then this model and that of Kyle (1985) are equivalent in the sense that the equilibrium values of market depth, the expected total trading volume, and the expected price level are the same in both models.

Equivalence was proved for two separate cases. In both, the linear equilibrium always exists in the hedger model. In the first case, the uninformed traders' risk-aversion parameter has to equal a specific value which is determined by the variance of the asset value and the endowment, the information precision, and the number of hedgers and informed traders. In the second case, equivalence holds for a limit economy where there are an infinite number of infinitely risk-averse hedgers but the total amount of noise in the economy is finite.

APPENDIX

Proof of Proposition 1. First, I show that for the given value of R the linear equilibrium always exists in the hedger model. Substitute for R on the left-hand side of (1) to obtain

$$\frac{2(2-t)(1+Q)\sqrt{k\Sigma_v}}{\sqrt{mt\Sigma_w}} > \frac{2\sqrt{mt\Sigma_v}}{\sqrt{k\Sigma_w}}.$$

The inequality holds because $(1 + Q) = (2 - t + mt) > mt$, $k \geq 1$, and $t \leq 1$.

Second, substitute the given value of R into (4). This shows that $D = -1$. If, in addition, $k\Sigma_w = \Sigma_u$ then $\Gamma_h = \Gamma_n$ from Lemmas 1 and 2. $E(|y_n|) = E(|y_h|)$ and $E(|P_n| = E(|P_h|)$ follow from the previous two results, given the assumption that $k\Sigma_w = \Sigma_u$.

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